Complex Analysis

October 25, 2022

Problem 1. Find all complex roots of $z^{17} = -1$.

Problem 2. Show that e^z satisfies the Cauchy-Riemann equations.

Problem 3. Prove or disprove that $|z|^2$ is holomorphic in $\mathbb{C}$.

Problem 4. Show that the function $f(z) = (|z|)^2$ is smooth as a function of real variables x, y, i.e., all the mixed partial derivatives of any order exist, however, it is not complex differentiable at any point different from the origin.

Problem 5. Let $f : \Omega \to \mathbb{C}$ be a complex differentiable function. Derive the Cauchy-Riemann equation.

Problem 6. Let $W = \mathbb{C} \setminus \{x + \sqrt{-1}y \in \mathbb{C} : x \leq 0, y = 0\}$. Define $\text{Log} : W \to \mathbb{C}$ by

$$\text{Log}(z) = \log r + \sqrt{-1}\theta,$$

where $z = re^{\sqrt{-1}\theta}$ for $-\pi < \theta < \pi$.

(1) $\text{Log}z$ is holomorphic.

(2) $e^{\text{Log}z} = z$

(3) $(\text{Log}z)' = 1/z$ for every $z \in W$.

Problem 7. (1) Show that there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = x^2 + y^2 \quad \frac{\partial f}{\partial y} = 2xy$$

(2) Show that the condition ”simply-connectedness” in Poincaré lemma is necessary.

Problem 8. Let $U = \mathbb{C} \setminus \{0\}$ and $F : U \to \mathbb{C}$ be a holomorphic function defined by $F(z) = 1/z$. Prove or disprove that there exists a holomorphic function $G : U \to \mathbb{C}$ such that $G'(z) = F(z)$.

Problem 9. Compute the following:

$$(1) \quad \int_{|z-i|=\frac{1}{4}} \frac{1}{1+z^2} dz \quad \text{and} \quad (2) \quad \int_\gamma \left(\bar{z}^2 + z - \frac{1}{z} \right) dz$$

where γ is given by $\gamma(t) = e^{it}$ for $0 \leq t \leq \pi/2$.

Problem 10. Compute the following:

$$\int_\gamma z^m dz$$

where $m \in \mathbb{Z}$ and γ is given by $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$.

Problem 11. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function. Prove that f is a constant function.

Problem 12. Prove that if $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)|^2 \leq C(1+|z|^2)^N$ then f is a polynomial in z, of degree at most N.

Problem 13. Let $f(z) = z^N + a_{N-1}z^{N-1} + \cdots + a_0$. Show that $f(z) = 0$ has exactly N zeros counting orders without using fundamental theorem for Algebra.

Problem 14. Let f be an entire function. For $z \in \mathbb{C}$, define $\gamma_z : [0,1] \to \mathbb{C}$ by

$$\gamma_z(t) = tz.$$

Define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f(z)dz.$$

Prove that F is an entire function.

Problem 15. Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain D. Fix a point $p \in D$ and define the function $F : D \to \mathbb{C}$ by

$$F(z) = \int_{\gamma} f(z)dz$$

for some curve γ from p to z. Show that F is a well-defined holomorphic function and $F'(z) = f(z)$.

Problem 16. Let f be a holomorphic function on a domain containing the closed the unit disc $\mathbf{D}$. With the usual notation $z = x + \sqrt{-1}y, w = u + \sqrt{-1}v$, show that

$$f(z) = \frac{1}{\pi}\int_{\mathbf{D}} \frac{f(w)}{1 - z\bar{w}}dudv$$

for every z with $|z| < 1$.

Problem 17. Compute $\displaystyle\int_0^\infty \frac{1}{1+x^4}dx.$

Problem 18. Compute $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^3}dx.$

Problem 19. Compute $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2}dx.$

Problem 20. Compute $\displaystyle\int_0^\infty \frac{\log x}{3+x^2}dx.$

Problem 21. Show that the Laurent expansion of $f(z) := e^{\frac{1}{z}}$ converges on $\mathbb{C}^*$, and find its Laurent expansion.

Problem 22. Let $f(z) = \frac{1}{(z-1)(z-2)}$. Find three Laurent series of f centered at 0.

Problem 23. Find two Laurent series of

$$f(z) = \frac{1}{z^2(z-2)}$$

centered at 0.

Problem 24. Let $f : \mathbb{C} \to \mathbb{C}$ be a entire function which satisfies the following:

$$\lim_{|z| \to \infty} |f(z)| = \infty.$$

Prove that f is a polynomial with a finite degree.

Problem 25. Prove that if $f : D^*(P, r) \to \mathbb{C}$ has an essential singularity at p, then there exists a sequence $\{z_n\} \in D^*(P, r)$ with $\lim_{n \to \infty} z_n = P$ and

$$|(z_n - p)^n f(z)| \geq n.$$

Problem 26. Let $f : U \setminus \{P\} \to \mathbb{C}$ be a holomorphic function which has a pole at P. Then what kind of singularity does $\frac{1}{f}$ have at P? Justify your answer.

Problem 27. Suppose that D is a bounded domain and $\{f_j\}$ is a sequence of nowhere vanishing holomorphic functions on D which converges uniformly on every compact subsets of D to a limit function f.

(1) Prove that $\{f_j'\}$ converges uniformly to f' on every compact subsets of D.

(2) Show that f is nowhere vanishing or $f \equiv 0$.

Problem 28. Suppose that a holomorphic function f is defined on an open set containing the closed unit disc $\overline{\mathbf{D}} := \{z : |z| \leq 1\}$ which never vanishes in $\mathbf{D}$ and satisfies that $|f(z)| = 3$ for all z with $|z| = 1$. Show that f is a constant function.

Problem 29. Prove that if $f : U \to \mathbb{C}$ is holomorphic, $z_0 \in U$, and $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0.

Problem 30. Let $f : D \to \mathbb{C}$ be a holomorphic function such that $f'(z_0) \neq 0$ for some $z_0 \in D$. Prove that f is locally one-to-one near z_0.

Problem 31. Prove the Normal Form Theorem by following steps.

(1) If $g : \mathbf{D} \to \mathbb{C}^*$ is holomorphic then $g = e^h$ for some holomorphic function h on $\mathbf{D}$. (Hint: Consider the function $H(z) := \int_0^z \frac{g'(w)}{g(w)} dw$ and Problem 9.)

(2) If f is holomorphic near p and $k = Ord_p(f)$ then there is a neighborhood U of p and an injective holomorphic function $g : U \to \mathbb{C}$ such that

$$f(g^{-1}(w)) = w^k$$

Problem 32. Let $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic function such that $f(0) = 0$.

(1) Show that $|f(z)| \leq |z|$ for $z \in \mathbf{D}$.

(2) Show that $|f'(0)| \leq 1$.

(3) Suppose that there exists a point $z_0 \in \mathbf{D}$ such that $|f(z_0)| = |z_0|$. Show that $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Problem 33. Let $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic function. Show that for $a \in \mathbf{D}$ and $b = f(a)$,

$$|f'(a)| \leq \frac{1-|b|}{1-|a|}.$$

Problem 34. Let $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic function. By using the Schwarz Lemma and automorphisms of the unit disc, prove that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

for $z \in \mathbf{D}$. Prove also that if the equality holds, then f is an automorphism.

Problem 35. Let $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic function such that $f(0) = 0$.

(1) Show that $|f(z)| \leq |z|$ for $z \in \mathbf{D}$.

(2) Show that $|f'(0)| \leq 1$.

(3) Suppose that there exists a point $z_0 \in \mathbf{D}$ such that $|f(z_0)| = |z_0|$. Show that $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Problem 36. Prove that

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

Problem 37. Let $a \in \mathbf{D}$ and let

$$L(z) = \frac{z-a}{1-\bar{a}z}.$$

Define $L_1 = L$, and, for $j \geq 1$, $L_{j+1} = L \circ L_j$. Prove that $\lim L_j$ exists, uniformly on compact subsets of $\mathbf{D}$, and determine what holomorphic function it is.

Problem 38. Let $D \subset \mathbb{C}$ be a bounded domain and let $\{f_j\}$ be a sequence of holomorphic functions on D. Assume that

$$\int_D |f_j|^2 \, dxdy < C < +\infty,$$

where C does not depend on j. Prove that $\{f_j\}$ is a normal family.

Problem 39. Let $U := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane. Prove that

$$\mathrm{Aut}(U) = \left\{ w = \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ with } ad - bc = 1 \right\}.$$

Problem 40. Let Ω be a bounded domain in $\mathbb{C}$ and let $f \in \mathrm{Aut}(\Omega)$. Suppose that $f(P) = P$ and $f'(P) = 1$. Prove that f is the identity.

Problem 41. Let Ω be a domain in $\mathbb{C}$ and let $u : \Omega \to \mathbb{R}$ be a harmonic function. Prove that if u^2 is also harmonic, then u is constant.

Problem 42. Let Ω be a bounded domain in $\mathbb{C}$. Let $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$ which is nonconstant.

(1) Prove that there exists $z_0 \in \partial\Omega$ such that

$$|f(z_0)| > |f(z)| \quad \text{for all} \quad x \in \Omega.$$

(2) Prove that either f has a zero in Ω or $|f|$ has a minimum value on $\partial\Omega$.

(3) If $f(z) = u(x, y)+iv(x, y)$, then show that v has a maximum and minimum only on the boundary $\partial\Omega$ not in Ω.

Problem 43. Prove the maximum principle for harmonic functions as follows.

(1) Prove that the real part of a holomorphic function $f = u + iv : U \to \mathbb{C}$ is harmonic.

(2) Let $u : \Delta \to \mathbb{R}$ be a harmonic function in the unit disc Δ. Prove that there exists a harmonic function $v : \Delta \to \mathbb{R}$ such that $u + iv$ is holomorphic.

(3) Let u be a harmonic function in $U \in \mathbb{R}^2 \equiv \mathbb{C}$. Prove that if there exists a point (x_0, y_0) such that

$$u(x_0, y_0) = \sup_{(x,y)\in U} u(x, y),$$

then u is a constant function.

Problem 44. Prove the Poisson integral formula by the following steps.

(1) Let $f : D_1 \to D_2$ be a holomorphic map and $u : D_2 \to \mathbb{R}$ be a harmonic function. Show that $u \circ f$ is harmonic.

(2) Prove the Mean value property:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt.$$

(3) Let u be a harmonic function defined on $\mathbf{D}$. Using an automorphism of $\mathbf{D}$, prove the following

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it}) \cdot \frac{1 - |z|}{|z - e^{it}|}dt.$$

Problem 45. Let $F : \mathbf{D}^* \to \mathbb{C}$ be a holomorphic function. Prove that the function

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta$$

for $0 < r < 1$, is constant independent of r.

Problem 46. Let $\{u_n\}$ be a sequence of harmonic functions on a domain D in $\mathbb{C}$. Suppose that $u_n(z) \le u_{n+1}(z)$ for all $z \in D$. If $u_n(z_0) \to \infty$ as $n \to \infty$ for some $z_0 \in D$, prove that $u_n \to \infty$ uniformly on compact subsets. More precisely, prove that given a compact subset $K \subset D$ and a real number $M > 0$ there exists $N \in \mathbb{N}$ such that

$$u_n(z) > M \quad \text{for all } \; z \in K \; \text{ and } \; n > N.$$