함수해석학 Sample problems

Problem 1. Let f be a bounded linear functional on a subspace Z of a normed space X. Then prove that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X satisfying $\|\tilde{f}\|_X = \|f\|_Z$.

Problem 2. Let X be a normed space.

(a) Let x_0 be any nonzero element of X. Prove that there exists a bounded linear functional g on X such that $\|g\| = 1$ and $g(x_0) = \|x_0\|$.

(b) If x_0 is such that $f(x_0) = 0$ for all $f \in X'$, show that $x_0 = 0$.

Problem 3. Prove that every Hilbert space is reflexive.

Problem 4. Let (x_n) be a sequence in a norm space X.

(a) Suppose that (x_n) converges weakly to x. Then prove that the weak limit x of (x_n) is unique. Moreover, show that the sequence $(\|x_n\|)$ is bounded.

(b) If $\dim X < \infty$, show that weak convergence implies strong convergence.

Problem 5. Let $T : \mathcal{D}(T) \subset X \longrightarrow Y$ be a linear operator where X, Y are normed spaces. Prove or give a counterexample to each of the following statements:

(a) If the operator T is closed, then it is bounded.

(b) If the operator T is bounded, then it is closed.

Problem 6. Show that the dual space of ℓ^p is ℓ^q where $1 < p < \infty$ and q is the conjugate of p that is $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 7. Show that

$$d(f, g) = \int_a^b |f(t) - g(t)|\, dt$$

defines a metric on the set

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continous } \}.$$

Problem 8. Let X be the set of all ordered n-tuples of real numbers and $x = (\xi_1, \cdots, \xi_n),\ y = (\eta_1, \cdots, \eta_n) \in X$. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined by

$$\|x\|_1 = \sum_{i=1}^{n} |\xi_i| \text{ and } \|x\|_2 = \left(\sum_{i=1}^{n} |\xi_i|^2 \right)^{\frac{1}{2}}$$

satisfy

$$\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1,$$

Problem 9. Let s be the set of all (bounded or unbounded) sequences of complex numbers.

(a) Prove that

$$d(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|} \quad \text{for } x = (x_j),\ y = (y_j) \in s$$

is a metric, that is, (s,d) is a metric space.

(b) Prove that not every metric on a vector space can be obtained from a norm by using the metric d in (a).

Problem 10. Prove that the ℓ^p space with $1 \le p < \infty$ is separable.

Problem 11. Let X be the set of all continuous real-valued functions on $J = [a,b]$, and $d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$ for $x, y \in X$. Then show that the metric space (X,d) is complete.

Problem 12. Let X be the space of all ordered n-tuples $x = (\xi_1, \cdots, \xi_n)$ of real numbers and and $d(x,y) = \max_j |\xi_j - \eta_j|$ for $y = (\eta_j) \in X$. Show that the metric space (X,d) is complete.

Problem 13. Let X be the set of all positive integers and and $d(m,n) = |m^{-1} - n^{-1}|$. Show that the metric space (X,d) is not complete.

Problem 14. Show that if a subspace Y of a metric space consists of finitely many points, then Y is complete.

Problem 15. If the metric space (X,d) is complete, prove that $(X,\tilde{d})$, where $\tilde{d} = \frac{d}{1+d}$ is complete.

Problem 16. Show that $\{x_1, \cdots, x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in the space $C[a,b]$.

Problem 17. Let $(X, \|\cdot\|)$ be a normed space. Show that the closed unit ball $\bar{B}_1(0) = \{x \in X \mid \|x\| \le 1\}$ is convex.

Problem 18. If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

Problem 19. Prove that in a Banach space, an absolutely convergent series is convergent.

Problem 20. Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for ℓ^p where $1 \le p < \infty$.

Problem 21. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on X, prove that the Cauchy sequences in $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are the same.

Problem 22. Let $X = C[0,1]$ and $Y = \text{span}(x_0, x_1, \cdots)$, where $x_j(t) = t^j$, so that Y is the set of all polynomials. Show that Y is not closed in X.

Problem 23. Let $T : \mathcal{D}(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1, \cdots, x_n\}$ is a linearly independent set in $\mathcal{D}(T)$, prove that the set $\{Tx_1, \cdots, Tx_n\}$ is linearly independent.

Problem 24. Let X and Y be normed spaces. Show that a linear operator $T : X \to Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

Problem 25. If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have that the strict inequality $\|Tx\| < \|T\|$.

Problem 26. Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\|x\| \text{ for all } x \in X$$

show that $T^{-1} : Y \to X$ exists and is bounded.

Problem 27. Show that the functionals defined on $C[a,b]$ by

$$f_1(x) = \int_a^b x(t)y_0(t)\,dt \quad \text{for } y_0 \in C[a,b] \text{ and } f_2(x) = \alpha x(a) + \beta x(b) \quad \text{for } \alpha, \beta \text{ fixed}$$

are linear and bounded.

Problem 28. The null space $N(M^*)$ of a set $M^* \subset X^*$ is defined to be the set of all $x \in X$ such that $f(x) = 0$ for all $f \in M^*$. Show that $N(M^*)$ is a vector space.

Problem 29. Let $M \neq 0$ be any subset of a normed space X. The annihilator M^a of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M. Thus M^a is a subset of the dual space X' of X. Show that M^a is a vector subspace of X' and is closed. What are X^a and $\{0\}^a$?

Problem 30. If z is any fixed element of an inner product space X, show that $f(x) =< x, z >$ defines a bounded linear functional f on X, of norm $\|z\|$.

Problem 31. Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \to \|x\|$ and $< x_n, x > \to < x, x >$ imply convergence $x_n \to x$.

Problem 32. Show that the annihilator $M^\perp$ of a set $M \neq \emptyset$ in an inner product space X is a closed subspace of X.

Problem 33. Show that in a Hilbert space H, convergence of $\sum \|x_j\|$ implies convergence of $\sum x_j$.

Problem 34. Show that ℓ^∞ and ℓ^1 are normed linear spaces.

Problem 35. A functional T on a normed linear space X is said to be Lipschitz provided there is a $c \geq 0$ such that

$$|T(g) - T(h)| \leq c\|g - h\| \text{ for all } g, h \in X.$$

The infimum of such c's is called the Lipschitz constant for T. Show that a linear functional is bounded if and only if it is Lipschitz, in which case its Lipschitz constant is $\|T\|$, where $\|T\|$ is the norm of T.

Problem 36. For X and Y normed linear spaces and $T \in B(X,Y)$, show that $\ker T$ is a closed subspace of X and that T is one-to-one if and only if $\ker T = \{0\}$. Here, $B(X,Y)$ is the collection of bounded linear operators from X to Y.

Problem 37. Let T be a linear operator from a normed linear space X to a finite-dimensional normed linear space Y. Show that T is continuous if and only if $\ker T$ is a closed subspace of X.

Problem 38. Let X be a finite dimensional normed linear space of dimension n and X^* be a dual space of X.

(a) Let $\{e_1, \cdots, e_n\}$ be a basis for X. For $1 \leq i \leq n$, define $\psi_i \in X^*$ by $\psi_i(x) = x_i$ for $x = x_1e_1 + \cdots + x_ne_n \in X$. Show that $\{\psi_1, \cdots, \psi_n\}$ is a basis for X^*. Thus $\dim X = n$.

(b) Show that the natural embedding $J : X \to X^{**}$ is one-to-one. Then use (a) to show that $J : X \to X^{**}$ is onto, so X is reflexive.

Problem 39. For each point x in a normed linear space X, show that

$$\|x\| = \sup\{\psi(x) | \psi \in X^*, \|\psi\| \leq 1\}.$$

Problem 40. Let S be a subset of a Hilbert space H. Show that $S = S^{\perp\perp}$ if and only if S is a closed subspace of H.