Exam Topics. Homotopy and homotopy equivalences, Simplicial cohomology, Singular homology, Exact sequences of homology groups. Excision, Mayer–Vietoris sequences, Degree, CW complexes, Cellular homology, Homology with coefficients, Universal coefficients theorem, Cohomology groups, Cup product, Cohomology rings.

Problem 1. Consider the following letters of alphabets.

A B C D E F O P R T Z

1. Classify the above letters of alphabets up to *homeomorphisms*. Namely, group the above letters which are homeomorphic to each other.
2. Classify the above letters of alphabets up to *homotopy equivalences*. Namely, group the above letters which are homotopy equivalent to each other.

Problem 2. If "yes", prove the statement. If "no", provide a counterexample.

1. Is the compactness invariant under a homotopy equivalence?
2. Is the connectedness invariant under a homotopy equivalence?

Problem 3. If a continuous function $f\colon X \to S^n$ is *not* surjective, then show that f is null-homotopic, that is, f is homotopic to a constant map.

Problem 4. Compute the homology groups of the following space.

1. the torus $T^2 = S^1 \times S^1$,
2. a punctured torus $T^2 - \{p\}$,
3. a twice punctured torus $T^2 - \{p, q\}$ where p and q are distinct points in T^2,
4. the Klein bottle K,
5. the real projective space $\mathbb{R}P^2$,
6. the closed orientable surface $T^2 \# T^2$ of genus 2.

Problem 5.
1. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups in all dimensions.
2. Are these spaces homeomorphic?

Problem 6. Let X be the space obtained from $\mathbb{R}^3$ removing the three coordinate axes,

$$X = \mathbb{R}^3 - \left(\{(x, 0, 0) \mid x \in \mathbb{R}\} \cup \{(0, y, 0) \mid y \in \mathbb{R}\} \cup \{(0, 0, z) \mid z \in \mathbb{R}\}\right).$$

Compute the homology groups $H_*(X)$ of X.

Problem 7. Compute the homology groups of the space X obtained from the solid torus by identifying the boundary torus, that is,

$$X := S^1 \times D^2 / \sim$$

where $x \sim y \Leftrightarrow x = y$, or both x and y are in $S^1 \times \partial D^2$.

Problem 8. Let $f\colon (X, A) \to (Y, B)$ be a continuous function such that both $f\colon X \to Y$ and its restriction $f\colon A \to B$ are homotopy equivalences. Prove that $f_*\colon H_n(X, A) \to H_n(Y, B)$ is an isomorphism for each n.

Problem 9. Let n and m be distinct natural number. Let U (resp. V) be a nonempty open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Show that U and V are *not* homeomorphic.

Problem 10. Let D^n be the n-dimensional ball and S^{n-1} the $(n-1)$-dimensional sphere, realized as the boundary of D^n.

1. Prove that there is no retraction $D^n \to S^{n-1}$.
2. Prove that every continuous map $D^n \to D^n$ has a fixed point.

Problem 11.

1. Show that $H_0(X, A) = 0$ if and only if A meets each path-connected component of X.
2. Show that $H_1(X, A) = 0$ if and only if $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A

Problem 12.

1. For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusion $i_\alpha\colon X_\alpha \to \bigvee_\alpha X_\alpha$ induces an isomorphism
$$\bigoplus_\alpha i_{\alpha,*}\colon \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right), \tag{1}$$
provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.
2. Compute the homology groups of the figure eight shape $S^1 \vee S^1$ using (1).

Problem 13. For the case of the inclusion $f\colon (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs – there is no $g\colon (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity through maps of pairs.

Problem 14. Let X be the Möbius strip,
$$X := ([0,1] \times [0,1]) / \sim$$
where $\sim$ is the equivalence relation generated by $(x, 0) \sim (1-x, 1)$ for each $x \in [0,1]$. Let Y be the cylinder,
$$Y := ([0,1] \times [0,1]) / \sim$$
where $\sim$ is the equivalence relation generated by $(x, 0) \sim (x, 1)$ for each $x \in [0,1]$. Verify that X and Y are *not* homeomorphic, following the outline below.

1. Compute the local homology group $H_2(X, X - \{x\})$ for $x \in \partial X$.
2. Compute the local homology group $H_2(X, X - \{x\})$ for $x \in X - \partial X$.
3. If there is a homeomorphism $f\colon X \to Y$, then $f|_{\partial X}\colon \partial X \to \partial Y$ is a homeomorphism.
4. Conclude that X and Y are not homeomorphic.

Problem 15. Let SX be the suspension of a topological space X. Show that
$$\widetilde{H}_n(X) \simeq \widetilde{H}_{n+1}(SX) \quad \text{for all } n.$$

Problem 16. Using the Mayer–Vietoris sequence, compute the homology group of the Klein bottle K.

Problem 17. Let $f\colon S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

Problem 18. Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$.

Problem 19. Let
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$
Let $S^2 \to S^2$ be a continuous map such that
$$\|f(\mathbf{x}) - \mathbf{x}\| < 1$$
for all $\mathbf{x} \in S^2$. Show that f is surjective.

Problem 20. A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\widehat{f}\colon S^2 \to S^2$. Show that the degree of $\widehat{f}$ is equal to the degree of f as a polynomial.

Problem 21.

1. For finite CW complex X and Y, show that
$$\chi(X \times Y) = \chi(X) \times \chi(Y).$$
2. If a finite CW complex X is the union of subcomplexes A and B, show that
$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Problem 22.

1. Compute the homology group of the real projective space $\mathbb{R}P^n$.
2. Compute $H_i(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$ by cellular homology, using the standard CW structures on $\mathbb{R}P^n$ with $\mathbb{R}P^m$ as its m-skeleton.

Problem 23. The closed orientable surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a compact topological space X. Compute the homology groups of X via the Mayer–Vietoris sequence for this decomposition of X into two copies of R.

Problem 24. For X a finite CW complex and $\mathbb{F}$ a field, show that the Euler characteristic $\chi(X)$ can also be computed by the formula
$$\chi(X) = \sum_n (-1)^n \dim H_n(X;\mathbb{F}),$$
the alternating sum of the dimensions of the vector spaces $H_n(X;\mathbb{F})$.

Problem 25.

1. Compute $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.
2. Let A be an abelian group. Show that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \mathrm{Ker}(\cdot n\colon A \to A)$.

Problem 26. Let K be the Klein bottle.

1. Compute the homology groups $H_n(K;\mathbb{Z}/8\mathbb{Z})$.
2. Compute the homology groups $H_n(K;\mathbb{Z}/3\mathbb{Z})$.

Problem 27. Show that if $\widetilde{H}_n(X;\mathbb{Q}) = 0$ and $\widetilde{H}_n(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for all n and all primes p, then $\widetilde{H}_n(X;\mathbb{Z}) = 0$ for all n.

Problem 28. Recall from Proposition 22 that the homology groups of $\mathbb{R}P^n$ are
$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & k \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$
Compute the cohomology groups $H^k(\mathbb{R}P^n;\mathbb{Z})$.

Problem 29. Suppose that a topological space X has the following homology groups
$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 3 \\ \mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/7\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n = 2 \text{ or } n \geq 4. \end{cases}$$
Compute the cohomology groups $H^*(X)$ of X (with $\mathbb{Z}$-coefficient).

Problem 30. Show that if $f\colon S^n \to S^n$ has degree d, then
$$f^*\colon H^n(S^n;G) \to H^n(S^n;G)$$
is multiplication by d.

Problem 31. For a topological space X, let

$$\langle\ ,\ \rangle\colon C^n(X) \times C_n(X) \to \mathbb{Z}$$

by the given by the paring $\langle \psi, \alpha \rangle = \psi(\alpha)$. In terms of this pairing, notice that the coboundary map

$$\delta\colon C^n(X) \to C^{n+1}(X)$$

is defined by $\langle \delta(\psi), \alpha \rangle = \langle \psi, \partial\alpha \rangle$ for all $\alpha \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle\ ,\ \rangle\colon H^n(X;\mathbb{Z}) \times H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$

Problem 32.

1. Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients.
2. Do the same for the Klein bottle.

Problem 33. Show that

1. $H^*(\mathbb{R}P^{2k};\mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$ where $|\alpha| = 2$.
2. $H^*(\mathbb{R}P^{2k+1};\mathbb{Z}) \simeq \mathbb{Z}[\alpha, \beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$ where $|\alpha| = 2$ and $|\beta| = 2k + 1$.

Problem 34. Suppose that $k > 0$ and $\ell > 0$. Using cup products, show that every continuous function $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism

$$H_{k+\ell}(S^{k+\ell}) \to H_{k+\ell}(S^k \times S^\ell).$$

Problem 35. Use cup products to compute the map

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$$

induced by the map that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, $(z_0, z_1, \ldots, z_n) \mapsto (z_0^d, z_1^d, \ldots, z_n^d)$, for a fixed integer $d > 0$.

Problem 36. Consider two spaces $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$.

1. Show that their cohomology rings

$$H^*(\mathbb{R}P^3;\mathbb{Z}) \simeq H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z})$$

   are isomorphic (as rings).
2. Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are not homotopy equivalent.

Problem 37. Show that there does not exist a continuous function

$$f\colon S^4 \to \mathbb{C}P^2$$

such that

$$f^*\colon H^4(\mathbb{C}P^2) \simeq \mathbb{Z} \to H^4(S^4) \simeq \mathbb{Z}.$$

maps a generator of $H^4(\mathbb{C}P^2)$ into a generator of $H^4(S^4)$.