MA73259 Riemannian Geometry (Practice problems for QE)

1. Let M be a connected n-dimensional smooth manifold.

   (a) Write the definition of a Riemannian metric on M.

   (b) For the Riemannian metric $g = \frac{4}{(1+|x|^2)^2} dx_i \otimes dx_i$ on $\mathbb{R}^n$, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, find the length of the curve $\sigma : [0,1] \to \mathbb{R}^n$, $\sigma(t) = (t, 0, \ldots, 0)$.

2. Let M be a smooth manifold (Hausdorff with countable basis) whose differentiable structure is $\{(U_\beta, x_\beta)\}$.

   (a) Prove that there exists a Riemannian metric on M.

   (b) Write the definition of affine connection ∇.

   (c) Prove that given a Riemannian manifold (M, g) there exists a unique affine connection ∇ on M satisfying the symmetric property and compatibility with the Riemannian metric g.

3. Let (M, g) be a Riemanninan manifold, X a vector field on M, and ω_X the dual 1-form of X. Show that $d\omega_X(Y, Z) = g(\nabla_Y X, Z) - g(Y, \nabla_Z X)$.

4. Show that the metric in coordinates satisfies:

   (a) $\partial_s g^{ij} = g^{ik} \partial_s g_{kl} g^{lj}$

   (b) $\partial_s g^{ij} = -g^{il} \Gamma^j_{sl} - g^{jl} \Gamma^i_{sl}$

5. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a geodesic.

   (a) Using a system of coordinate (U, x) about $\gamma(t_0)$, where $t_0 \in I$, write the geodesic equation.

   (b) Write the definition of the exponential map.

   (c) Prove that exponential map is a local diffeomorphism.

6. Let (M, g) be a Riemannian manifold.

   (a) Write the definition of normal coordinate at p.

   (b) For any point $p \in M$, show that there exists a normal coordinate at p.

7. Consider the upper half space

   $$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2; y > 0\}$$

   Define a metric on $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = 0$.

   (a) Calculate the Christoffel symbol.

(b) Let $v_0 = (0,1)$ be a tangent vector at $(0,1) \in \mathbb{R}^2_+$. And $v(t)$ is a vector field generated from v_0 via parallel transport along the curve $c : [0,1] \to \mathbb{R}^2_+$, $c(t) = (t,1)$. Prove that the angle between $v(t)$ and the y-axis equals to t with respect to the metric g.

8. Prove that the isometries of $S^n \subset \mathbb{R}^{n+1}$, with the induced metric, are the restrictions to S^n of the linear orthogonal maps of $\mathbb{R}^{n+1}$.

9. Prove the Riemannian geometry fundamental theorem: Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:

   (a) $\nabla_X Y - \nabla_Y X = [X,Y]$ for all $X, Y \in TM$.

   (b) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, for all $X, Y, Z \in TM$

10. Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_o,t} : T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_0,t}(v), v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

11. Let $\tilde{g} = e^{2\psi} g$ be a metric conformally equivalent to g. Show that

$$\tilde{\nabla}_X Y = \nabla_X Y + \left((\nabla_X \psi) Y + (\nabla_Y \psi) X - g(X,Y)\nabla\psi \right).$$

12. Let M^2 be an isometric immersed hypersurface in $\mathbb{R}^3$ and $c : I \to M$ is a smooth curve on M. Let $V : I \to \mathbb{R}^3, V(t) \in T_{c(t)}M$ be a tangential vector field on c, defined by.

   (a) Prove $V(t)$ is parallel if and only if $\frac{dV}{dt}$ is normal to $T_c(t)M \subset \mathbb{R}^3$, where $\frac{dV}{dt}$ is a normal derivation in $\mathbb{R}^3$.

   (b) Let S^2 be a unit sphere in $\mathbb{R}^3$. Show that the velocity field along great circles, parametrized by arc length, is a parallel field.

13. Consider the following conditions for a smooth function $f : (M,g) \to \mathbb{R}$ on a connected Riemannian manifold:

   (a) $|\nabla f|$ is constant

   (b) $\nabla_{\nabla f} \nabla f = 0$

   (c) $|\nabla f|$ is constant on the level sets of f.

   Show that $(a) \Leftrightarrow (b) \Rightarrow (c)$ and give an example to show that the last implication is not a bi-implication.

14. Write the definition of Riemannian curvature $R : TM \times TM \times TM \times TM \to \mathbb{R}$, and prove that

(a) $R(X, Y, Z, W) = -R(Y, X, Z, W)$

(b) $R(X, Y, Z, W) = -R(X, Y, W, Z)$

(c) $R(X, Y, Z, W) = R(Z, W, X, Y)$

15. Prove the first Bianchi identity:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

where $X, Y, Z, W, T \in TM$.

16. Prove the second Bianchi identity:

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W).$$

where $X, Y, Z, W, T \in TM$.

17. Let X be a Killing field on a Riemannian manifold M. (that is, for any $Y, Z \in TM$, $\langle \nabla_Y X + \nabla_Z X \rangle = 0$. Define a mapping $A_X : TM \to TM$ by $A_X(Z) = \nabla_Z X$ for $Z \in TM$. Consider the function $f : M \to \mathbb{R}$ given by $f(q) = \langle X, X \rangle_q, q \in M$. Let $p \in M$ be a critical point of f (that is, $df_p = 0$). Prove that for any $Z \in TM$, at p.

(a) $\langle A_X(Z), X \rangle(p) = 0$.

(b) $\langle A_X(Z), A_X(Z) \rangle(p) = \frac{1}{2} Z_p(Z\langle X, X \rangle) + \langle R(X, Z)X, Z \rangle$.

18. Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive. Prove that every Killing vector field X on M has a singularity (that is, there exists $p \in M$ s.t. $X(p) = 0$).

19. Let X be a *Killing field* on (M, g) and consider the function $f = \frac{1}{2}g(X, X) = \frac{1}{2}|X|^2$. Compute that

(a) $\nabla f = -\nabla_X X$,

(b) $\text{Hess} f(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V)$,

(c) $\Delta f = |\nabla X|^2 - Ric(X, X)$.

Hint: A vector field X is a *Killing field* if $\mathcal{L}_X g = 0$.

20. (Schur's Lemma) Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is isotropic, that is, for each $p \in M$. the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_p M$. Prove that M has constant sectional curvature, that is, $K(p, \sigma)$ also does not depend on p.

21. Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 and ∇^2 be the Levi-Civita connection of M_1 and M_2, respectively.

(a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by $\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2$, where $X_1,\ Y_1 \in TM_1$ and $X_2, Y_2 \in TM_2$.

(b) For every $p \in M_1$, the set $(M_2)_p = \{(p,q) \in M_1 \times M_2 : q \in M_2\}$ is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

(c) Let $\sigma(x, y) \subset T(p,q)(M_1 \times M_2)$ be a plane such that $x \in T_pM_1$ and $y \in T_qM_2$. Show that $K(\sigma) = 0$.

22. Show that $x : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$x(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi), \ (\theta, \phi) \in \mathbb{R}^2.$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3(1) \subset \mathbb{R}^4$, whose image $x(\mathbb{R}^2)$ is a torus T^2 with sectional curvature zero in the induced metric.

23. Let $x : \mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an immersion of the torus T^2 into S^3 given by

$$x(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi), \ (\theta, \phi) \in \mathbb{R}^2.$$

Show that x is a minimal immersion.

24. Let $\Sigma \subset (M, g)$ be a submanifold. Let ∇^Σ denote the connection on Σ that comes from the metric induced by g. Define the second fundamental form of Σ in M by

$$\mathrm{II}(X, Y) = \nabla_X Y - \nabla^\Sigma_X Y$$

(a) Show that $\mathrm{II}(X, Y)$ is symmetric,

(b) Show that $\mathrm{II}(X, Y)$ is always normal to Σ,

(c) Show that $\mathrm{II} = 0$ on Σ iff Σ is totally geodesic.

25. Consider the upper half-space

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2; y > 0\}$$

with the Riemannian metric given by

$$g_{11} = 1, \ g_{12} = 0, \ g_{22} = \frac{1}{y}.$$

Show that the length of the vertical segment

$$x = 0, \ \varepsilon \le y \le 1, \ \text{with } \varepsilon > 0$$

tend to 2 as $\varepsilon \to 0$, conclude from this fact that such a metric is not complete.

26. Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty.

27. Let M be a complete simply-connected Riemannian manifold with nonpositive sectional curvature (i.e. $K(\sigma, p) \leq 0$ for all $p \in M$ and for all $\sigma \subset T_p(M)$). Prove that M is diffeomorphic to $\mathbb{R}^n$.

28. Let M^n be a complete Riemannian manifold. Suppose that the Ricci curvature of M satisfies

$$Ric_p(v) \geq \frac{1}{r^2} > 0,$$

for all $p \in M$ and for all $v \in T_p(M)$. Show that M is compact and the diameter $diam(M) \leq \pi r$.

29. Prove the Rauch comparison theorem:

Assume $K_M \leq a$, where K_M is a sectional curvature.

(a) If J is a Jacobi field along a unit-speed geodesic $\gamma|_{[0,l]}$ and J is perpendicular to γ, then

$$||J||^{''} + a||J|| \geq 0$$

along γ.

(b) If ϕ is a solution on $[0, l]$ of $\phi^{''} + a\phi = 0$, $\phi(0) = ||J||(0)$, $\phi^{'}(0) = ||J||^{'}(0)$, and $\phi \neq 0$ on $(0, l)$, then $\left(\frac{||J||}{\phi}\right)^{'} \geq 0$ and $||J|| \geq \phi$ on $(0, l)$.