Problems for Qualifying Exam (Commutative Algebra)

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R is always a commutative ring with unity 1 unless a special comment.

Rings and Ideals

1. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some positive integer n.

   (1) Show that the set $N(R)$ of nilpotent elements in R is an ideal in R (which is called the *nilradical of* R).

   (2) Let $u \in R$ be a *unit*, that is, there is a multiplicative inverse $v \in R$ such that $uv = vu = 1$. Show that $a + u$ is also a unit when $a \in N(R)$ is nilpotent.

   (3) Show that $N(R)$ equals to the intersection of all prime ideals in R.

   (4) Let F be a field, and let $R = F[x]$ be a ring of polynomials over F. What are the nilradical $N(R)$ and the Jacobson radical $J(R)$, which is the intersection of all maximal ideals in R?

2. Let $f_1, \cdots, f_r \in R$ be elements such that the ideal $(f_1, \cdots, f_r) = (1) = R$ is improper. Show that the ideal $(f_1^{n_1}, \cdots, f_r^{n_r})$ is also improper for any positive integers $n_1, \cdots, n_r$.

3. Let R be a Noetherian ring, that is, any ascending chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

   eventually terminates. Show that any surjective ring homomorphism $\varphi : R \to R$ is injective (**Hint**: consider a chain of ideals $\ker \varphi \subseteq \ker \varphi^2 \subseteq \ker \varphi^3 \subseteq \cdots$).

4. Find all the prime and maximal ideals in each of the following rings:

(1) $\mathbb{R}[x]$; (2) $\mathbb{R}[[x]]$; (3) $\mathbb{C}[x]$; (4) $\mathbb{C}[[x, y, z]]$.

5. Let k be a field.

   (1) Show that $k[x, y]/(y^2 - x^3) \simeq k[t^2, t^3]$ are isomorphic.

   (2) Show that $k[t^2, t^3] \subseteq k[t]$ is not isomorphic to $k[t]$.

Modules

6. (1) Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

   (2) Show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Conclude that $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}$.

   (3) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

7. Let k be a field, and let $R = k[x, y, z]$ be a polynomial ring. Show that the following sequence

$$0 \to R \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} R \to M \to 0$$

   is a projective resolution of the R-module $M = R/(x, y, z) \simeq k$. Such a projective resolution is called a *Koszul resolution*.

8. Let k be a field, and let $V_1, \cdots, V_n$ be k-vector spaces. Suppose that there is an exact sequence

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0.$$

   Show that the alternating sum of dimensions $\sum_{i=1}^{n} (-1)^i \dim V_i = 0$ is zero.

9. Let m, n be positive integers, and let k be a field. Compute and simplify each of the following tensor products of modules.

(1) $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$;

(2) $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$;

(3) $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$;

(4) $k[x]/(x^n) \otimes_{k[x]} k[x]/(x^m)$;

(5) $k[x] \otimes_k k[x]$;

(6) $k[x] \otimes_{k[x]} k[x]$.

10. Let m, n be positive integers, and let k be a field. Compute and simplify each of the following Tor modules for every $i \in \mathbb{Z}$. Note that a field k can be seen as a $k[x]$-module, or a $k[x, y]$-module via the quotient by the ideal (x), or (x, y).

(1) $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$;

(2) $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$;

(3) $\mathrm{Tor}_i^{\mathbb{Q}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q})$;

(4) $\mathrm{Tor}_i^{k[x]}(k, k)$.

(5) $\mathrm{Tor}_i^{k[x]}(k, k[x]/(x^n))$.

(6) $\mathrm{Tor}_i^{k[x,y]}(k, k)$.

11. Let m, n be positive integers, and let k be a field. Compute and simiplify each of the following Hom modules.

(1) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$;

(2) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q})$;

(3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$;

(4) $\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m))$;

(5) $\mathrm{Hom}_k(k[x], k[x])$;

(6) $\mathrm{Hom}_{k[x]}(k[x], k[x])$.

12. Let m, n be positive integers, and let k be a field. Compute and simplify each of the following Ext modules for every $i \in \mathbb{Z}$.

(1) $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$;

(2) $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q})$;

(3) $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$;

(4) $\mathrm{Ext}^i_{k[x]}(k[x]/(x^n), k[x]/(x^m))$;

(5) $\mathrm{Ext}^i_k(k[x], k[x])$;

(6) $\mathrm{Ext}^i_{k[x]}(k[x], k[x])$.

13. Let $(R, \mathfrak{m})$ be a local ring, and let M, N be finitely generated R-modules. Show that if $M \otimes_R N = 0$, then either $M = 0$ or $N = 0$ (**Hint**: use Nakayama's lemma on $M/\mathfrak{m}M$).

14. Let R be an integral domain (so that the zero ideal (0) is prime)

(1) What is the localization $R_{(0)}$ with respect to the zero ideal (0)?

(2) Show that the localization $R_{\mathfrak{p}}$ with respect to a prime ideal $\mathfrak{p} \subset R$ is a subring of the quotient field $K(R)$ of R.

15. Let R be an integral domain. Compute

$$\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}, \qquad \text{and} \qquad \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$$

where the intersection is taken over all the prime ideals $\mathfrak{p} \subset R$, and over all the maximal ideals $\mathfrak{m} \subset R$.

16. Let R be an integral domain, and let $f_1, \cdots, f_r \in R$ be a set of nonzero elements such that $(f_1, \cdots, f_r) = (1)$. Show that $R_{f_1} \cap R_{f_2} \cap \cdots \cap R_{f_r} = R$.

Factorizations and Algebraic Integers

17. A proper ideal $I \subset R$ is called *primary* if

$$ab \in I \text{ implies either } a \in I \text{ or } b^n \in I \text{ for some } n > 0,$$

in other words, I is primary iff $R/I \neq 0$ and every nonzero divisor in R/I is nilpotent.

(1) Find all primary ideals in $\mathbb{Z}$.

(2) Let $R = k[x, y]$ be a polynomial ring over k with two indeterminates x, y. Find a primary ideal I such that

$$(x, y)^2 \subsetneq I \subsetneq (x, y).$$

(3) Show that a power of a prime ideal $\mathfrak{p}^n$ needs not to be primary using the following counterexample. Let $R = k[x, y, z]/(xy - z^2)$ and let $\overline{x}, \overline{y}, \overline{z}$ be images of x, y, z in R. Check that $\mathfrak{p} = (\overline{x}, \overline{z})$ is prime, but $\mathfrak{p}^2$ is not primary.

18. Find a primary decomposition of each of the following ideals:

(1) $I = (x^2, xy)$ in $k[x, y]$;

(2) $I = (xz - y^2, yw - z^2)$ in $\mathbb{C}[x, y, z, w]$.

19. (1) Explain why the ring $D = \mathbb{Z}[\sqrt{-5}]$ is not a UFD.
    (2) Show that D is a Dedekind domain.
    (3) Find a factorization of the ideal $(6) \subset D$ as a product of prime ideals in D.
    (4) Explain why the ring $\mathbb{Z}[\sqrt{5}]$ is not a Dedekind domain.
    (5) Show that the ring $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ is a Dedekind domain.

Others

20. Let $p > 0$ be a prime number, and let $\mathbb{Z}_p$ be the ring of p-adic integers.
    (1) Explain why $\mathbb{Z}_p$ is a ring properly contains $\mathbb{Z}$ as a subring.
    (2) Find all the prime ideals in $\mathbb{Z}_p$.
    (3) Explain why $\mathbb{Z}_p$ is an uncountable set.
    (4) Show that the series $1 + 2 + 4 + 8 + 16 + \cdots$ converges to -1 in the ring of 2-adic integers $\mathbb{Z}_2$, using the 2-adic norm $\|\cdot\|_2$.

21. Let $p = 5$ and $\alpha = 3/250 \in \mathbb{Q}$. Find the 5-adic expression of α in $\mathbb{Q}_p$ (= the field of fraction of $\mathbb{Z}_p$).

22. Let $R = k[x]$ be a polynomial ring over a field k, and let $I = (x)$. Describe the ring of formal completion $\hat{R}$ of R with respect to I.

23. Let V be a finite dimensional vector space over a field k, and let $V^* = \mathrm{Hom}_k(V, k)$ be the dual vector space of V.
    (1) Show that $V^* \otimes V$ can be identified with the space of linear transformations $\varphi : V \to V$.
    (2) We define the trace map $tr : V^* \otimes V \to k$ as

$$tr(\sum_{i=1}^{n} c_i(\varphi_i \otimes v_i)) := \sum_{i=1}^{n} c_i \varphi_i(v_i).$$

    Explain why the above definition of tr coincides with the usual trace of a linear transformation (or equivalently, the trace of a square matrix).

(3) Let $\varphi : V \to V$ be a linear transformation. For each $0 \le i \le \dim V$, we define $\wedge^i(\varphi)$ as

$$\wedge^i(\varphi)(v_1 \wedge \cdots \wedge v_i) := \varphi(v_1) \wedge \cdots \wedge \varphi(v_i)$$

and extend it linearly. Show that there is a constant $c = c(\varphi) \in V$ such that $\wedge^{\dim V}(\varphi)$ is same as the multiplication map by c.

(4) Explain why the above constant $c(\varphi)$ is same as the determinant $\det(\varphi)$.

24. Let k be a field, and let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over k with n indeterminates. What is the module of Kähler differential $\Omega_{R/k}$?

25. Let $V = \mathbb{R}^2$ be 2-dimensional vector space over V, and let $\{e_1, e_2\}$ be its standard ordered basis. Consider the W-state tensor

$$W = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 \in V \otimes V \otimes V.$$

Explain why its tensor rank is 3 and its border rank is 2.