PROBLEM BANK : SEVERAL COMPLEX VARIABLES

1. Let Δ^2 be the unit polydisc in $\mathbb{C}^2$. Prove that a holomorphic function $f : \Delta^2 \to \mathbb{C}$ satisfies that

$$f(z,w) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1|=r_1} \int_{|\zeta_2|=r_2} \frac{f(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - w)} d\zeta_1 d\zeta_2,$$

for some $r_1, r_2 > 0$.

2. Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain and let $p \in \Omega$. If a holomorphic map $f : \Omega \to \Omega$ satisfies

- $f(p) = p$, and
- $df(p) = \mathrm{Id}$,

then f is the identity.

3. Let $\Omega \in \mathbb{C}^n$ be a holmorphically convex domain. Prove that Ω is Hartogs pseudoconvex.

4. Prove that $B^2 := \left\{z \in \mathbb{C}^2 : |z| < 1\right\}$ is a domain of holomorphy by the definition of domain of holomorphy.

5. Prove that $D := \left\{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\right\} \setminus \left\{(z,w) \in \mathbb{C}^2 : 1/2\,|z| < 1, |w| < 1/2\right\}$ is not a domain of holomorphy.

6. Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary which admits a smooth defining function φ satisfying

- $\varphi \in C^\infty(\Omega)$.
- $\varphi = 0$ on $\partial\Omega$.
- $d\varphi \neq 0$ on $\partial\Omega$.
- $\left(\dfrac{\partial^2}{\partial z^j \partial \overline{z^k}} \varphi\right)$ is positive-definite in $\overline{\Omega}$.

Prove that there exists a smooth defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ for Ω so that ρ is strictly plurisubharmonic in Ω.

7. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Prove that $V(f) = \{z \in \mathbb{C}^n : f(z) = 0\}$ is not bounded.

8. Prove the following

(1) Prove that $\Omega_1 \subset \mathbb{C}^2$ defined by

$$\Omega_1 = \left\{(z_1,z_2) \in \mathbb{C}^2 : \rho_1(z_1,z_2) = |z_1|^2 + |z_2|^2 + 1 - 2\,|z_1| - r^2 < 0\right\}$$

is Levi strongly pseudoconvex for $0 < r < 1$.

(2) Prove that $\Omega_2 \subset \mathbb{C}^2$ defined by

$$\Omega_2 = \left\{(z_1,z_2) \in \mathbb{C}^2 : \rho_1(z_1,z_2) = |z_1|^2 + |z_2|^2 + 1 - 2\sqrt{x_1^2 + x_2^2} - r^2 < 0\right\}$$

is Levi pseudoconvex for $0 < r \leq 1/2$ and Levi strongly pseudoconvex for $0 < r < 1/2$.

9. Let $\Omega\left\{(z,w)\in\mathbb{C}^2 : |z|<|w|<1\right\}$. Prove that Ω is a domain of holomorphy. Also prove that if U is a sufficiently small neighborhood of $\overline{\Omega}$, then U is not a domain of holomorphy.

10. Let Δ^2 is the polydisc in $\mathbb{C}^2$. Compute the automorphism group Δ^2 which is defined as follows:

$$\mathrm{Aut}(\Delta^2)=\left\{f:\Delta^2\to\Delta^2 : f \text{ is a biholomorphism}\right\}.$$

11. Prove that the unit polydisc Δ^2 and the unit ball B^2 in $\mathbb{C}^2$ are not biholomorphic.

12. Let $A^{p,q}(\Omega)$ be the space of smooth (p,q)-forms in Ω. Consider the following sequence.

$$A^{p,q}(\Omega)\to A^{p,q+1}(\Omega)\to A^{p,q+2}(\Omega),$$

where each map is given by $\overline{\partial}$. Prove that $\overline{\partial}\circ\overline{\partial}=0$.

13. Let V_1 and V_2 be Hilbert spaces and let $T:V_1\to V_2$ be a (possibly unbounded) linear operator. State the definition of $\psi\in\mathrm{Dom}(T^*)$. Prove that if $g\in\mathrm{Dom}(T^*)$, then there exists a unique element $T^*g\in V_1$ such that

$$\langle f,T^*g\rangle_1=\langle Tf,g\rangle_2 \quad\text{for}\quad f\in\mathrm{Dom}(T).$$

14. Let $A^{p,q}(\Omega)$ be the space of smooth (p,q)-forms in Ω. Consider the following sequence.

$$A^{0,0}(\Omega)\to A^{0,1}(\Omega)\to A^{0,2}(\Omega),$$

where each map is given by $\overline{\partial}$. For $l=0,1,2$, φ_l is a smooth function in Ω and $A^{0,l}(\Omega)$ is equipped with the following hermitian product.

$$\langle f,g\rangle_l=\int_\Omega f\cdot g e^{-\varphi_l}dV,$$

where $\cdot$ is given by the pointwise product of forms. Compute the formal adjoint of

$$\overline{\partial}:A^{0,0}(\Omega)\to A^{0,1}(\Omega),$$

and

$$\overline{\partial}:A^{0,1}(\Omega)\to A^{0,2}(\Omega).$$