1. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere.
   (a) Show that $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a smooth manifold.
   (b) Show that $f : S^2 \to \mathbb{R}$ defined by $f(x, y, z) = z$ is smooth at $(0, 0, 1)$.
   (c) Show that $A : S^2 \to S^2$ defined by $A(x, y, z) = (-x, -y, -z)$ is smooth at $(0, 0, 1)$.
2. Let $X$ be the set of all straight lines in the plane. Show that $X$ is a smooth manifold.
3. Consider the real projective space $\mathbb{RP}^4 = (\mathbb{R}^5 \setminus \{0\})/\sim$, where $(x_1, \cdots, x_5) \sim (y_1, \cdots, y_5)$ if there exists $a \in \mathbb{R} \setminus 0$ such that $x_i = ay_i$ for all $i$.

   For each $i \in \{1, \cdots, 5\}$, let $U_i = \{[x_1 : \cdots : x_5] \in \mathbb{RP}^4 \mid x_i \neq 0\}$ and define a map $\phi_i : U_i \to \mathbb{R}^4$ by

   $$\phi_i([x_1 : \cdots : x_5]) = \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_5}{x_i}\right).$$

   Compute the transition map $\phi_4 \circ \phi_3^{-1}$.
4. Let us consider the two smooth structures on the real line $\mathbb{R}$: $(\mathbb{R}, x_1)$, where $x_1 : \mathbb{R} \to \mathbb{R}$ is given by $x_1(x) = x$; $x_2 : \mathbb{R} \to \mathbb{R}$ is given by $x_2(x) = x^3$.
   (a) Show that the maximal structures determined by $(\mathbb{R}, x_1)$ and $(\mathbb{R}, x_2)$ are distinct.
   (b) Show that the differentiable structure determined by $(\mathbb{R}, x_1)$ and $(\mathbb{R}, x_2)$ are diffeomorphic. (Hint: the mapping $f : (\mathbb{R}, x_1) \to (\mathbb{R}, x_2)$ given by $f(x) = x^3$ is diffeomorphsm)
5. Consider the real projective space $\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\})/\sim$, where $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if there is $0 \neq \lambda \in \mathbb{R}$ such that $a_i = \lambda\, b_i$ for all $i$.

   Let $(U, \phi)$ be a coordinate chart, where $U = \{[z_1 : z_2 : z_3] \in \mathbb{RP}^2 \mid z^1 \neq 0\}$ and $\phi : U \to \mathbb{R}^2$, $\phi([z_1 : z_2 : z_3]) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right)$. Let $p = [1 : 1 : 2]$.
   (a) Find the inverse $\phi^{-1}$ of $\phi$.
   (b) Let $f : \mathbb{RP}^2 \to \mathbb{R}$ be a function $f([z_1 : z_2 : z_3]) = \dfrac{(z_2)^2}{(z_1)^2 + (z_3)^2}$. Find $f \circ \phi^{-1}$.
   (c) Compute $\left(\dfrac{\partial f}{\partial x_1}\right)_p$, where $\phi(m) = (x_1(m), x_2(m))$ for $m \in U$.
6. Let $M$ and $N$ be smooth manifolds. Prove that $M \times N$ is a smooth manifold.
7. Let $M_1$ and $M_2$ be smooth manifolds. Prove that the projection map $\pi_i : M_1 \times M_2 \to M_i$ is a smooth map.
8. Let $M$ be a smooth manifold and let $p$ be a point in $M$. How is a tangent vector to $M$ at $p$ defined?
9. Let $M$, $N$, and $P$ be manifolds and let $F : M \to N$ and $G : M \to P$ be smooth maps. Let $p \in M$. Prove the following

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.

(c) $d(\mathrm{Id_M})_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

(d) If $F$ is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

10. Let $F : M = \mathbb{R}^3 \to N = \mathbb{R}^3$ be a map

$$F(x_1, x_2, x_3) = (3x_1x_2 + 2x_1x_3 + x_2x_3, x_1 + x_2 + x_3, x_1^2 + x_2^2 + x_3^2)$$

for $(x_1, x_2, x_3) \in M$. Let $(y_1, y_2, y_3) \in N$ denote a point in $N$. Compute $dF(\frac{d}{dx_1}), dF(\frac{d}{dx_2})$, and $dF(\frac{d}{dx_3})$.

11. Prove that the tangent bundle of an $n$-dimensional smooth manifold is also a smooth manifold of dimension $2n$.

12. Let $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2$. Prove that $F^{-1}(100)$ is an embedded submanifold of $\mathbb{R}^2$.

13. The Lie bracket of vector fields $X$ and $Y$ is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

(a) Show that $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for three smooth vector fields $X$, $Y$, and $Z$.

(b) For 1-form $\omega$, the following identity holds

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Verify this formula by computing each side for $\omega = fdg$ where $f$ and $g$ are smooth functions.

14. Let $X = x\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial z}$ and $Y = z\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} + yz\dfrac{\partial}{\partial z}$ be vector fields on $\mathbb{R}^3$. Compute the Lie bracket of $X$ and $Y$. That is, compute $[X, Y]$.

15. Let $M = \mathbb{R}^2$. Let $X$ be a vector field on $M$ defined by $X(x, y) = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$.

(a) Fine the flow of $X$.

(b) Let $\omega = dx + dy$. Compute $\mathcal{L}_X\omega$, the Lie derivative of $\omega$ with respect to $X$.

16. Let $\alpha = (3x_1 + 5x_2 + 2_3)\, dx_1 + x_1x_2x_3\, dx_2 + e^{x_1+x_2+x_3}\, dx_3$ be a differential 1-form on $\mathbb{R}^3$. Compute $d\alpha$, the differential of $\alpha$.

17. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a map $F(x_1, x_2, x_3) = (x_1x_2x_3, x_1x_2 + x_1x_3 + x_2x_3)$. Let $x_1$, $x_2$, $x_3$ be coordinates of $\mathbb{R}^3$, and let $y_1$, $y_2$ be coordinates of $\mathbb{R}^2$. Compute $F^*(dy_1)$ and $F^*(dy_2)$, the pull-backs by $F$ of $dy_1$ and $dy_2$.

18. If $M$ is a compact smooth manifold, every $C^1$ map $f : M \to \mathbb{R}$ has at least two critical points.

19. Show that the special linear group $SL(n, \mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$.

20. Define $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $f(X) = XX^t$, where $X$ is an $n \times n$-square matrix and $X^t$ is its transpose.

    (a) $f(X)$ is symmetric.

    (b) $Df(X)M = XM^t + MX^t$.

    (c) If $f(X) = I_n$, then $Df(X)$ maps $M_n(\mathbb{R})$ onto the space of symmetric matrices. (Hint: Given a symmetric matrix $S$, consider $M = \frac{1}{2}SX$.)

21. Let $GL_n(\mathbb{R})$ be the set of all invertible matrices. Let $f(X) = X^{-1}$.

    (a) Justify that $GL_n(\mathbb{R})$ is a smooth manifold.

    (b) $Df(X)M = -X^{-1}MX^{-1}$

22. Suppose $E$ is a smooth vector bundle over $M$. Show that the projection map $\pi : E \to M$ is a surjective smooth submersion.

23. Let $M = \mathbb{R}^2 \setminus \{0\}$, let $\omega$ be a 1-form on $M$ given by

$$\omega = \frac{xdy - ydx}{x^2 + y^2},$$

    and let $\gamma : [0, 2\pi] \to M$ be the curve defined by

$$\gamma(t) = (\cos t, \sin t).$$

    (a) Show that $\omega$ is closed.

    (b) Compute $\int_\gamma \omega$.

    (c) Justify that $\omega$ is not exact.

24. Let $\omega = x\, dy \wedge dz + 3y\, dz \wedge dx$ be a differential 2-form on $\mathbb{R}^3$. Compute $\int_{S^2} \omega$ with your chosen orientation on $S^2$, where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$.

25. Consider the 2-dimensional sphere $S^2$

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$$

    with induced orientation from $\mathbb{R}^3$. Compute the following integral over $S^2$

$$\int_{S^2} x_2\, dx_1\, dx_3.$$