편미분방정식론 Sample problems

* Let $U \subset \mathbb{R}^n$ be a bounded open subset with a smooth boundary ∂U and fix a time $T > 0$. As usual set $U_T = U \times (0, T]$ and $\Gamma_T = \bar{U}_T - U_T$. All given functions are assumed smooth, unless otherwise stated.

Problem 1. Find an explicit formula for a solution u of the following initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Problem 2. Assume $f \in C^1(\mathbb{R}^n \times \mathbb{R})$ and $g \in C^1(\mathbb{R}^n)$. Solve the following nonhomogeneous problem

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $b \in \mathbb{R}^n$.

Problem 3. Prove that if U is connected and $u \in C^2(U) \cap C(\overline{U})$ is harmonic in U, then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

by using the mean-value property.

Problem 4. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant: that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox), \quad x \in \mathbb{R}^n$, then $\Delta v = 0$.

Problem 5. Prove the uniqueness of solutions for the following Dirichlet problem

$$-\Delta u = f \text{ in } U, \quad u = g \text{ on } \partial U$$

by using the energy method.

Problem 6. Prove that the n-dimensional Laplace equation is invariant under all rigid motions.

Problem 7. Let

$$I(w) := \int_U \frac{1}{2}|Dw|^2 - fw \, dx$$

where $w \in \mathcal{A} := \{w \in C^2(\bar{U}) : w = g \text{ on } \partial U\}$. Prove that a function $u \in C^2(\bar{U})$ solves the Dirichlet problem $-\Delta u = f \ \text{ in } U, \quad u = g$ on ∂U if and only if $I(u) = \min_{w \in \mathcal{A}} I(w)$.

Problem 8. Suppose that $u \in C^2(\overline{U})$ satisfies that $-\Delta u \geq 0$ in U.

(a) Prove that

$$u(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy \quad \text{for all } B(x,r) \subset U.$$

(b) Using (a), prove that

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

Problem 9. Assume that U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \text{ a.e. in } U.$$

Then show that u is constant a.e. in U.

Problem 10. Without using the mean-value property, prove that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic in U, then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Problem 11. Let U be a bounded open subset of $\mathbb{R}^n$ and fix a time $T > 0$. Suppose that $u, v \in C^2(\overline{U}_T)$ solve

$$u_t - \Delta u = 0 \text{ in } U_T, \quad u = g \text{ on } \partial U \times [0,T]$$

and

$$v_t - \Delta v = 0 \text{ in } U_T, \quad v = g \text{ on } \partial U \times [0,T]$$

for some function g. Show that if $u(x,T) = v(x,T)$ for $x \in U$, then $u \equiv v$ in U_T.

Problem 12. Prove that there exists at most one solution $u \in C^2(\bar{U}_T)$ of the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } U \times \{t = 0\} \end{cases}$$

by using the energy method.

Problem 13. Solve

$$au_x + bu_y + cu = 0 \text{ in } \mathbb{R}^2, \quad u(x,0) = g(x) \text{ for } x \in \mathbb{R}.$$

Here $a, b, c \in R$ are nonzero constants and g is a given function.

Problem 14. Let u be smooth and solve the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show that $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Using (a), prove that $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Problem 15. If $u \in C^2(U)$ satisfies

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, dS$$

for each ball $B_r(x) \subset U$, prove that u is harmonic.

Problem 16. If $u \in C(U)$ satisfies the mean-value property

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, dS = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) \, dy$$

for each ball $B_r(x) \subset U$, prove that $u \in C^\infty(U)$.

Problem 17. For a locally integrable function $f : U \to \mathbb{R}$, define its mollification

$$f^\epsilon := \eta_\epsilon * f \text{ in } U$$

where η_ϵ is the standard mollifier, that is, $\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right)$ for each $\epsilon > 0$. Here,

$$\eta(x) = C \exp\left(\frac{1}{|x|^2 - 1}\right) \text{ if } |x| < 1 \quad \text{and} \quad \eta(x) = 0 \text{ if } |x| \geq 1$$

for some constant $C > 0$ satisfying $\int_{\mathbb{R}^n} \eta dx = 1$. Prove that $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.

Problem 18. Prove that there exists a constant c, depending on n, U, such that

$$\max_{\bar{U}} |u| \leq c(\max_{\partial U} |g| + \max_{\bar{U}} |f|)$$

where u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

Problem 19. For each connected open set $V \subset\subset U$, prove that for any nonnegative harmonic function u in U,

$$\sup_V u \leq c \inf_V u$$

for some constant $c = c(n, V) > 0$.

Problem 20. Suppose that $u \in C^2(\overline{U})$ satisfies that $-\Delta u \leq 0$ in U.

(a) Prove that

$$u(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy \quad \text{for all } B(x,r) \subset U.$$

(b) Using (a), prove that

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Problem 21. Assume that $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show $u_t = u_{xx}$ if and only if

$$v'' + \frac{z}{2} v' = 0. \tag{1}$$

(b) Show that the general solution of (1) is

$$v(z) = c \int_0^z e^{-\frac{s^2}{4}} \, ds + d.$$

(b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$.

Problem 23. Given $g : [0,\infty) \to \mathbb{R}$ with $g(0) = 0$, derive that formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} e^{\frac{-x^2}{4(t-s)}} g(s) \, ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

Problem 24. Give a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(Hint: Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

Problem 25.

(a) Show that the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t,\ \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.